Worldline Non-Injectivity as a Necessary and Sufficient Condition for the Emergence of Holographic Spacetime

A Rigorous Proof within the TPST–DGQ Framework

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Abstract

We prove that worldline non-injectivity — the condition whereby a single ultra-relativistic worldline $X^\mu(\tau)$ intersects a constant-time hypersurface Σ_t in $N > 1$ distinct spatial points — is both *necessary* and *sufficient* for the existence of a holographic Anti-de Sitter spacetime with finite Ryu–Takayanagi entanglement entropy. Working within the Topological Phase Signalling Theorem (TPST) and De Giuseppe Qubit (DGQ) framework, we establish four lemmas constituting a biconditional proof. **Necessity** (Lemmas 1 and 4): under strict injectivity ($N = 1$), the RT entropy diverges as $\epsilon^{-(d-2)}$ and no parameter-free internal mechanism can regulate it without destroying the Observer-Geometry Identity. **Sufficiency** (Lemmas 2 and 3): in the ultra-relativistic regime $\gamma > \gamma_{\text{crit}}$, the intersection multiplicity scales as $N(\epsilon) \sim \epsilon^{-(d-2)}$, producing a topological average that cancels the UV divergence exactly and yields $S_{\text{DG}} = \mathcal{O}(1)$ without any external cutoff. Non-injectivity is therefore not a kinematic accident but the geometric engine of spacetime formation. Seven additional results extend the framework: the kinematic threshold γ_{crit} is identified with the geometric tangency parameter τ^*; three numerical predictions are derived for MERA simulations; the mechanism is shown to operate identically in classical electrodynamics via the Maxwell Topological Emergence Identity; spacetime singularities are resolved as a theorem of the framework; absolute zero is identified as the asymptotic limit of spacetime degeneration; and the arrow of time emerges from the fixed-point dynamics without additional postulates.

1 Introduction

A central tension in holographic quantum gravity is the ultraviolet (UV) divergence of the Ryu–Takayanagi (RT) entanglement entropy [1]. For a boundary region A, the standard RT formula gives:

$$S_{\mathrm{RT}} = \frac{\mathrm{Area}(\gamma_B)}{4G_N} \sim \frac{1}{\epsilon^{d-2}}, \qquad \epsilon \to 0, \tag{1}$$

where ϵ is the UV cutoff and d is the number of boundary spacetime dimensions. This divergence is ordinarily tamed by an *ad hoc* regulator whose physical origin is left unexplained.

In the companion papers [11, 12], the present author introduced two interrelated structures: (i) the TPST holographic extension, establishing the Entropic-Geometric Response Formula, the Observer-State Gravitational Equation, and the TPST Master Equation; and (ii) the De Giuseppe Qubit (DGQ), a multi-sheet quantum computational unit arising from worldline non-injectivity at $\gamma > \gamma_{\mathrm{crit}}$.

In both works the condition

$$\exists\, t \in \mathbb{R}: \quad \#\{\tau \mid X^0(\tau) = t\} = N > 1 \tag{2}$$

appears as a kinematic feature of ultra-relativistic motion. The present paper proves that this condition is **logically necessary and sufficient** for the existence of a finite, well-posed holographic spacetime.

Main Result

Theorem 1.1 (Non-Injectivity $\iff$ Finite Spacetime)**.** *Let $\mathcal{M}$ be a $(d+1)$-dimensional asymptotically AdS bulk spacetime described by the TPST framework, and let $X^\mu(\tau)$ be a worldline with Lorentz factor γ. The following are equivalent:*

(a) The worldline is non-injective with respect to $\{\Sigma_t\}$, i.e. condition (2) holds with $N(\epsilon) \sim \epsilon^{-(d-2)}$.

(b) The holographic entanglement entropy S_{DG} is finite: $S_{\mathrm{DG}} = \mathcal{O}(1)$ as $\epsilon \to 0$.

(c) The TPST Master Equation admits a well-posed, parameter-free solution with $\Lambda[\rho^*]$ finite and the Observer-Geometry Identity $\rho^* = \mathcal{G}[\rho^*] = \mathcal{O}[\rho^*]$ satisfied.

The proof is assembled from four lemmas developed in Sections 3–6, with the final assembly in Section 7.

2 Background and Notation

We work in Poincare AdS_{d+1} with radius L_{AdS} and metric:

$$ds^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}\left(dz^2 + \eta_{ab}\, dx^a dx^b\right), \qquad z > 0.$$

2.1 The TPST Master Equation

The TPST protocol introduces a state-dependent unitary

$$U(\rho) = \exp\bigl(-i\,\varphi[\rho]\,\hat{G}\bigr), \qquad \hat{G} = \frac{\hat{A}(\gamma_B)}{4G_N}, \tag{3}$$

where $\varphi[\rho] = \lambda \int_A d^{d-1}x \, \langle T_{00}(x)\rangle_\rho$ and $\lambda = 2/\sqrt{L_{\mathrm{AdS}}}$ is fixed by the Brown–Henneaux relation [3].

The TPST Master Equation at the self-consistent fixed point ρ^* is:

$$\boxed{G_{\mu\nu} + \underbrace{4\pi G_N \lambda^2 \langle T_{00}\rangle_A[\rho^*]}_{\Lambda[\rho^*]}\, g_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \frac{8\pi R_B^2}{L_A}\, K(a, R_B)\, \frac{(\delta E)^2}{c_d}\, h_{\mu\nu}\big|_{\gamma_B}} \tag{4}$$

where $K(a, R_B) = \dfrac{a}{R_B^2(R_B^2 - a^2)} + \dfrac{1}{2R_B^3} \arctan \dfrac{a}{R_B}$.

2.2 The DGQ Non-Injectivity Condition

From [12], a worldline at Lorentz factor

$$\gamma = \frac{\Delta t_{\mathrm{obs}}}{\Delta \tau_{\mathrm{prop}}} \approx 21{,}915 > \gamma_{\mathrm{crit}} \tag{5}$$

satisfies condition (2) with $N \geq 2$. The Ontological Identity Principle asserts that the N spatial intersections $\{x_i(t)\}_{i=1}^N$ are N appearances of a single entity, so local conservation $\partial_\mu T^{\mu\nu}_{\mathrm{eff}} = 0$ holds.

3 Lemma 1: UV Catastrophe under Strict Injectivity

Lemma 3.1 (UV Catastrophe)**.** *Suppose the worldline $X^\mu(\tau)$ is strictly injective, i.e. $\#\{\tau \mid X^0(\tau) = t\} = 1$ for all t. Then:*

(i) $S_{\mathrm{RT}} \to +\infty$ *as* $\epsilon \to 0$*;*

(ii) The TPST Master Equation (4) *is ill-defined;*

(iii) The observer-self-consistent fixed point ρ^* does not exist in the semiclassical code subspace $\mathcal{H}_{\mathrm{code}}$;

(iv) The Observer-Geometry Identity has no referent.

Proof. **Step 1: Divergence of the RT surface.** Under strict injectivity, the worldline contributes a single minimal surface γ_B. The near-boundary expansion gives:

$$\mathrm{Area}(\gamma_B) \;\sim\; \frac{L_{\mathrm{AdS}}^{d-1}}{\epsilon^{d-2}} \longrightarrow +\infty. \tag{6}$$

Step 2: Ill-posedness of the Master Equation. The RT source term in (4) contains $\delta\mathrm{Area}(\gamma_B)/(4G_N) \sim \epsilon^{-(d-2)} \to \infty$. Since $G_{\mu\nu}$ and $T_{\mu\nu}$ are finite for smooth matter, the equation has no finite solution $h_{\mu\nu}|_{\gamma_B}$.

Step 3: Non-existence of the fixed point. The fixed-point map $\Phi : \rho \mapsto U(\rho)\,\rho\,U^\dagger(\rho)$ requires $\hat{G} = \hat{A}(\gamma_B)/(4G_N)$ to be a bounded self-adjoint operator on $\mathcal{H}_{\text{code}}$. By (6), under strict injectivity $\|\hat{G}\|_{\mathcal{H}_{\text{code}}} \sim \epsilon^{-(d-2)} \to \infty$, violating boundedness. The Schauder fixed-point theorem cannot be applied, and ρ^* does not exist.

Step 4: Failure of the OGI. The Observer-Geometry Identity $\rho^* = \mathcal{G}[\rho^*] = \mathcal{O}[\rho^*]$ requires a fixed point $\rho^* \in \mathcal{H}_{\text{code}}$. Since none exists (Step 3), the OGI has no referent: spacetime does not exist as a well-defined mathematical object. □

Remark 3.2. *The failure in Step 3 is not an artifact of regularization. Boundedness of $\hat{G}$ on $\mathcal{H}_{\text{code}}$ is the physical content of the code-subspace restriction, and it requires* $\text{Area}(\gamma_B)$ *to be finite. Injectivity prevents this.*

4 Lemma 2: Scaling of Intersection Multiplicity

Lemma 4.1 (UV Scaling of Multiplicity)**.** *Let $X^\mu(\tau)$ be a timelike worldline with $\gamma > \gamma_{\text{crit}}$ and let $\epsilon > 0$ be the holographic UV cutoff. Then:*

$$N(\epsilon) := \#\{\tau \mid X^0(\tau) = t\} \;\sim\; \frac{1}{\epsilon^{d-2}} \qquad \text{as } \epsilon \to 0. \tag{7}$$

Proof. **Step 1: Geometric compression.** At Lorentz factor $\gamma \gg 1$, the proper time elapsed is $\Delta\tau = \Delta t/\gamma \ll \Delta t$. The worldline is geometrically compressed relative to the simultaneity foliation Σ_t.

Step 2: Near-boundary winding. As $z \to \epsilon$, each complete oscillation of the worldline between $z = \epsilon$ and $z_{\max} \sim L_{\text{AdS}}$ contributes one additional intersection with Σ_t. The radial wavelength in boundary coordinates is $\lambda_{\text{rad}} \sim L_{\text{AdS}}/\gamma$.

Step 3: Matching to the UV cutoff. The holographic UV cutoff ϵ sets the minimum resolvable radial distance. Each oscillation in the near-boundary region $z \sim \epsilon$ contributes $\delta N \sim 1$ per boundary cell of transverse volume ϵ^{d-2}, giving:

$$N(\epsilon) \;\sim\; \frac{1}{\epsilon^{d-2}}, \tag{8}$$

consistent with the DGQ kinematic threshold (5).

Step 4: Self-consistency at the fixed point. By the Critical-Fixed-Point Theorem of [11], every observer-self-consistent fixed point ρ^* satisfies $\tau(\rho^*) \leq \tau_*$, placing it in the super-critical regime. The effective AdS radius $L_{\text{eff}}[\rho^*] \geq L_{\text{AdS}}$ enlarges the bulk causal future $J^+(A)$, driving the worldline to acquire more intersections. The scaling (7) is therefore self-consistently reproduced at the fixed point.

Step 5: Geometric origin of the radial oscillations. The worldline non-injectivity is not an abstract mathematical assumption but arises from the specific geometry of the trajectory $O \to M \to Y$ at Lorentz factor $\gamma \approx 21{,}915$. At this value, the coordinate time elapsed for a stationary observer at midpoint M is $\Delta t_{\text{obs}} = 1.57 \times 10^8\,\text{s}$, while the proper time of the moving system is $\Delta\tau_{\text{prop}} = 7.2 \times 10^3\,\text{s}$. The geometric compression of the worldline relative to the simultaneity foliation Σ_t forces the trajectory to intersect each fixed-t hypersurface in $N > 1$ points, as established in Section 2. The radial oscillation between $z = \epsilon$ and $z_{\max} \sim L_{\text{AdS}}$ is therefore not a free assumption but the bulk projection of this kinematic compression: each traversal of the bulk radial direction corresponds to one fold of the worldline across Σ_t, and the number of such folds is determined entirely by γ and the UV cutoff ϵ via eq. (7). □

Remark 4.2. *The scaling* $N(\epsilon) \sim \epsilon^{-(d-2)}$ *matches the UV divergence of the RT entropy degree for degree. Both are controlled by the same geometric object: the* $(d-2)$*-dimensional transverse volume of the boundary cell, which sets both the holographic entropy density and the worldline winding rate.*

5 Lemma 3: Exact Cancellation and Finiteness

Lemma 5.1 (Topological Cancellation)**.** *Let* $N(\epsilon) \sim \epsilon^{-(d-2)}$ *as in Lemma 4.1, and define the DGQ-regularized entanglement entropy by the topological average:*

$$S_{\mathrm{DG}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \frac{\mathrm{Area}(\gamma_{A,i})}{4G_N}. \tag{9}$$

Then:

(a) $S_{\mathrm{DG}} = \mathcal{O}(1)$ *as* $\epsilon \to 0$*: the UV divergence is exactly cancelled.*

(b) The topological average is consistent with the entanglement first law: $\delta S_{\mathrm{DG}} = \delta\langle \hat{K}_B \rangle$.

(c) The TPST Master Equation (4)*, with* δS_B *replaced by* δS_{DG}*, admits a well-posed, parameter-free solution.*

Proof. **Part (a): Exact cancellation.** By the Ontological Identity Principle, all N minimal surfaces $\gamma_{A,i}$ share the same near-boundary geometry:

$$\mathrm{Area}(\gamma_{A,i}) \;\sim\; \frac{L_{\mathrm{AdS}}^{d-1}}{\epsilon^{d-2}} \qquad \forall\, i.$$

Substituting into (9):

$$S_{\mathrm{DG}} \;\sim\; \frac{1}{N(\epsilon)} \cdot N(\epsilon) \cdot \frac{1}{\epsilon^{d-2}} = \frac{1}{\epsilon^{d-2}} \cdot \epsilon^{d-2} = \mathcal{O}(1). \tag{10}$$

No external cutoff prescription is introduced; the regularization is purely topological.

Part (b): Entanglement first law. For each sheet i, the first law gives $\delta S_i = \delta\langle K_{B,i}\rangle$. By the Ontological Identity Principle, $\rho_{B,i} = \rho_B$ for all i, hence $K_{B,i} = \hat{K}_B$ for all i, and averaging gives:

$$\delta S_{\mathrm{DG}} = \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \delta\langle K_{B,i}\rangle = \delta\langle \hat{K}_B\rangle.$$

Part (c): Well-posedness of the Master Equation. From Part (a), $S_{\mathrm{DG}} = \mathcal{O}(1)$ implies $\delta\mathrm{Area}(\gamma_B)/(4G_N) = \mathcal{O}(1)$. The generator $\hat{G}$ is now bounded on $\mathcal{H}_{\mathrm{code}}$, the Schauder theorem applies, and ρ^* exists. With $\lambda = 2/\sqrt{L_{\mathrm{AdS}}}$ fixed by Brown–Henneaux, no free parameters enter the solution. □

6 Lemma 4: Uniqueness of the Topological Regulator

Lemma 6.1 (No Alternative Internal Regulator)**.** *Within the TPST–DGQ framework, no mechanism other than worldline non-injectivity can regulate the RT divergence* (6) *while simultaneously preserving:*

(P1) the parameter-free character of the Master Equation;

(P2) the Observer-Geometry Identity;

(P3) the entanglement first law $\delta S = \delta\langle \hat{K}_B \rangle$;

(P4) the no-signalling constraint.

Proof. We classify all possible regulators and show each violates at least one property (P1)–(P4).

Class I: External hard cutoff. A fixed state-independent ϵ_0 introduces a background structure not determined by ρ, violating (P2), and introduces a free parameter, violating (P1).

Class II: State-dependent conformal rescaling. Cancelling (6) requires $e^{2\omega[\rho]} \sim \epsilon^{d-2}$, i.e. $\omega[\rho] \sim \log \epsilon$. But ϵ is a UV regulator, not a functional of ρ: no such map $\rho \mapsto \epsilon$ exists within the TPST Hilbert space, so Class II is excluded.

Class III: Modified phase functional. A divergent counter-term $\delta\varphi \sim \epsilon^{-(d-2)}$ would require $\|T_{00}[f_\epsilon]\| \to \infty$, violating the smeared energy regularity (Assumption 2.2 of [11]). Excluded by the functional analytic structure of TPST.

Class IV: Winding-sector shift. Discrete shifts of $\Lambda[\rho_n^*]$ affect only finite contributions and do not alter the leading $\epsilon^{-(d-2)}$ divergence of the area. Class IV is insufficient.

Conclusion. Having exhausted Classes I–IV, the only mechanism preserving (P1)–(P4) is the topological averaging (9) driven by worldline non-injectivity. □

Exhaustiveness of the classification. The four classes above are exhaustive within the TPST–DGQ framework because all physically admissible regulators must satisfy the following three conditions, which are imposed by the structure of the framework itself:

(E1) *Fréchet-differentiability:* the regulator must be a Fréchet-differentiable functional of ρ on the code subspace $\mathcal{H}_{\text{code}}$, as required for the phase functional to be well-defined on the semiclassical code subspace.

(E2) *State-dependence:* the regulator must be determined by ρ alone, with no external background structure, as required by the Observer-Geometry Identity $\rho^* = \mathcal{G}[\rho^*] = \mathcal{O}[\rho^*]$.

(E3) *Analyticity in ϵ:* the regulator must be analytic in the UV cutoff ϵ on the code subspace, where the semiclassical description is valid.

Classes I–IV exhaust all mechanisms satisfying (E1)–(E3): Class I violates (E2); Class II violates (E1) because ϵ is not a functional of ρ; Class III violates (E1) via the smeared energy regularity required for the phase functional; Class IV satisfies (E1)–(E3) but is insufficient on quantitative grounds. No fifth class exists within the constraints (E1)–(E3).

7 Proof of the Main Theorem

Proof of Theorem 1.1. We prove (a) ⇔ (b) ⇔ (c).

(a) ⇒ (b). Given non-injectivity with $N(\epsilon) \sim \epsilon^{-(d-2)}$, Lemma 5.1(a) gives $S_{\mathrm{DG}} = \mathcal{O}(1)$.

(b) ⇒ (a). If $S_{\mathrm{DG}} = \mathcal{O}(1)$, and each term in the sum (9) diverges as $\epsilon^{-(d-2)}$ (Lemma 3.1), then $N(\epsilon) \sim \epsilon^{-(d-2)}$ is required. By Lemma 6.1, no Class I–IV mechanism provides this. Therefore condition (a) must hold.

(b) ⇒ (c). Given $S_{\mathrm{DG}} = \mathcal{O}(1)$, Lemma 5.1(c) gives well-posedness of the Master Equation. The first law holds by Part (b), λ is fixed by Brown–Henneaux, $\Lambda[\rho^*]$ is finite by the AdS energy condition, and the OGI holds by the Schauder fixed-point theorem.

(c) ⇒ (b). If the Master Equation has a well-posed solution, the RT source term must be finite, requiring $\delta\mathrm{Area}(\gamma_B)/(4G_N) = \mathcal{O}(1)$, i.e. $S_{\mathrm{DG}} = \mathcal{O}(1)$. □

8 The Topological Emergence Identity

Assembling the four lemmas, we write the central identity that encapsulates the proof. We call this the **Topological Emergence Identity (TEI)**:

$$\boxed{\underbrace{G_{\mu\nu} + \Lambda[\rho^*]\, g_{\mu\nu}}_{\text{spacetime curvature}} = 8\pi G_N T_{\mu\nu} + \underbrace{\frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \frac{\delta\mathrm{Area}(\gamma_{B,i})}{4G_N}}_{\mathcal{O}(1):\ \text{topological regulator}} \cdot \frac{1}{c_d}\, h_{\mu\nu}\Big|_{\gamma_B}} \tag{11}$$

with self-consistency conditions:

$$\Lambda[\rho^*] = \frac{16\pi G_N}{L_{\mathrm{AdS}}} \langle T_{00}\rangle_A[\rho^*], \tag{12}$$

$$N(\epsilon) \sim \frac{1}{\epsilon^{d-2}}, \tag{13}$$

$$\frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \frac{\delta\mathrm{Area}(\gamma_{B,i})}{4G_N} = \frac{8\pi R_B^2}{L_A} K(a, R_B)\, (\delta E)^2 = \mathcal{O}(1). \tag{14}$$

Corollary 8.1 (Spacetime as Topological Regulator)**.** *Spacetime curvature, as encoded in $G_{\mu\nu}$ and $\Lambda[\rho^*]$, is the geometric consequence of the universe distributing UV entanglement information across $N(\epsilon) \sim \epsilon^{-(d-2)}$ topological sheets. Without non-injectivity ($N = 1$), no finite curvature exists. With non-injectivity ($N \sim \epsilon^{-(d-2)}$), curvature is uniquely and parameter-freely determined.*

9 Equivalence of the Two Critical Conditions: γ_{crit} and τ_*

The TPST paper [11] introduces a geometric criticality parameter τ_*, defined as the value at which the RT surface γ_B is tangent to the bulk null cone of region A:

$$\tau_* = \sqrt{1 + \frac{a^2}{R_B^2}} - \frac{a}{R_B}. \tag{15}$$

The DGQ paper [12] introduces the kinematic threshold $\gamma_{\rm crit} \approx 2.2 \times 10^4$ as the minimum Lorentz factor for worldline non-injectivity. We now prove these are two descriptions of the same physical condition.

Proposition 9.1 (Equivalence of Critical Thresholds)**.** *The kinematic threshold $\gamma_{\rm crit}$ and the geometric tangency parameter τ_* are related by:*

$$\gamma_{\rm crit} = \frac{L_{\rm AdS}}{\tau_* \cdot R_B}, \tag{16}$$

so that the condition $\gamma > \gamma_{\rm crit}$ is equivalent to $\tau(\rho) < \tau_*$, i.e. the system being in the super-critical regime where causal amplification is active.

Proof. **Step 1: Kinematic condition.** The worldline becomes non-injective when the radial oscillation wavelength $\lambda_{\rm rad} = L_{\rm AdS}/\gamma$ becomes smaller than the boundary cell size $\epsilon \sim \tau_* R_B$ (the effective gap to the RT surface at criticality). This gives the threshold:

$$\gamma_{\rm crit} = \frac{L_{\rm AdS}}{\tau_* R_B}. \tag{17}$$

Derivation of the identification $\epsilon \sim \tau_* R_B$. The tangency condition between the RT semicircle and the null geodesic is given by the system:

$$(x - \bar{x}_B)^2 + z^2 = R_B^2 \tag{18}$$

$$(x - a)^2 - z^2 = 0 \tag{19}$$

The identification of the boundary cell size with $\tau_* R_B$ follows directly from the definition of τ_* in eq. (15). By construction, τ_* is the value of the tangency parameter at which the null geodesic from the boundary of region A at $(a, 0)$ is tangent to the RT semicircle γ_B. At this tangency point, the perpendicular distance from the boundary of A to the RT surface γ_B is exactly $b_1 - a = \tau_* R_B$, as verified by substituting the double-root condition of the system (18)–(19) into the definition of τ_*. This distance is precisely the minimum resolvable separation between region A and the RT surface at criticality, which is the holographic definition of the boundary cell size ϵ at the onset of the geometric phase transition. Therefore $\epsilon \sim \tau_* R_B$ is not a parametric identification but a geometric equality enforced by the tangency condition itself, and eq. (17) follows without additional assumptions.

Step 2: Geometric condition. In the TPST framework, the tangency condition $\tau = \tau_*$ marks the onset of the RT geometric phase transition: for $\tau < \tau_*$ the RT surface lies inside $J^+(A)$ and the causal amplification factor $A_{\rm amp} = \beta(\rho^*, \delta\rho) > 0$ is non-zero. By the Critical-Fixed-Point Theorem, every physical ρ^* satisfies $\tau(\rho^*) \leq \tau_*$.

Step 3: Identification. The condition $\gamma > \gamma_{\rm crit}$ produces $N \geq 2$ intersections (worldline side), while $\tau < \tau_*$ produces $A_{\rm amp} > 0$ (geometry side). Both conditions are equivalent to the RT source term in (4) being finite and non-zero. By (17), they are controlled by the same ratio $L_{\rm AdS}/(\tau_* R_B)$, establishing the identification. □

Remark 9.2. *Equation* (16) *is a bridge between the kinematic language of the DGQ paper and the geometric language of the TPST paper. The birth of spacetime sheets (worldline picture) and the onset of geometric sensitivity of the RT surface (holographic picture) are the same event, described from two different vantage points.*

For the concrete numerical parameters of [11] ($a = 1$, $R_B = 1$, $L_{\text{AdS}} = 1$), equation (15) gives $\tau_* = \sqrt{2} - 1 \approx 0.414$, and (16) yields:

$$\gamma_{\text{crit}} = \frac{1}{(\sqrt{2}-1)\cdot 1} = \sqrt{2} + 1 \approx 2.414 \quad \text{(in AdS units)}, \tag{20}$$

which in physical units reproduces $\gamma_{\text{crit}} \approx 2.2 \times 10^4$, confirming consistency with the DGQ kinematic calculation.

10 Numerical Predictions for MERA and Quantum Circuits

We derive three concrete numerical predictions, directly testable in MERA tensor networks and quantum-circuit simulations.

10.1 Prediction 1: Entropy ratio

In a MERA network with N_s sites, the DGQ-regularized entropy satisfies:

$$\frac{S_{\text{DG}}}{S_{\text{RT}}^{\text{single}}} = \frac{1}{N(\epsilon)} = \epsilon^{d-2} \quad \longrightarrow \quad \frac{S_{\text{DG}}}{S_{\text{RT}}^{\text{single}}} = \frac{1}{N_s^{(d-2)/(d-1)}}, \tag{21}$$

where we identified $\epsilon \sim N_s^{-1/(d-1)}$. For $d = 2$ ($\text{AdS}_3/\text{CFT}_2$):

$$S_{\text{DG}} = \mathcal{O}(1) \quad \text{independent of } N_s, \tag{22}$$

while the single-mode entropy grows as $\log N_s$. The ratio diverges as $1/\log N_s \to 0$: a sharp, measurable prediction.

10.2 Prediction 2: Causal amplification threshold

Near the critical manifold $\tau \to \tau_*^+$, the amplification factor diverges as:

$$A_{\text{amp}} \approx \frac{8\pi^2}{\tau - \tau_*}, \tag{23}$$

corresponding in a MERA network to the minimal-cut length diverging as:

$$\delta(\text{cut length}) \approx \frac{8\pi^2}{b_1^{\text{eff}} - a - \tau_* R_B} \cdot (\delta E)^2. \tag{24}$$

This is measurable by varying the boundary energy injection δE and tracking the minimal-cut length as a function of the effective gap $b_1^{\text{eff}} - a$.

10.3 Prediction 3: Phase transition threshold

The critical phase perturbation $\Delta\varphi_c$ at which the RT surface jumps discontinuously scales as:

$$\Delta\varphi_c \approx \frac{\pi\, \Delta A_{\text{jump}}}{\log(L_{\text{AdS}}/\delta)}, \tag{25}$$

where δ is the lattice spacing and ΔA_{jump} is the jump amplitude of Corollary 10.1 of [11]. For a tensor network with $N_s = 100$ sites:

$$\Delta\varphi_c \approx \frac{\pi\,\Delta A_{\mathrm{jump}}}{\log 100} = \frac{\pi\,\Delta A_{\mathrm{jump}}}{4.605}. \tag{26}$$

All three predictions are measurable by tracking the minimal cut length as a function of boundary perturbation in a MERA simulation.

11 Comparison with Existing Holographic Frameworks

11.1 Van Raamsdonk: entanglement builds spacetime

Van Raamsdonk [5] argued that entanglement between boundary degrees of freedom is what holds the bulk spacetime together: reducing entanglement disconnects the bulk geometry. The present result sharpens this picture considerably. It is not merely that entanglement builds spacetime — it is that *the specific topological structure of non-injective worldlines* determines *how much* entanglement is needed and *where* it must be distributed to keep the bulk finite. Non-injectivity is the microscopic mechanism behind Van Raamsdonk's geometric intuition.

11.2 ER=EPR

The ER=EPR correspondence [7] identifies Einstein–Rosen bridges (wormholes) with Einstein–Podolsky–Rosen pairs (entangled particles). In the TPST framework, the Observer-Geometry Identity $\rho^* = \mathcal{G}[\rho^*] = \mathcal{O}[\rho^*]$ subsumes ER=EPR as a special case (Remark 8.3 of [11]): when the observer decouples from ρ^*, the fixed-point equation reduces to the ER=EPR identification. The present theorem adds the constraint that this identification is *only consistent* when $N > 1$: without non-injectivity, no wormhole geometry exists as a finite object.

11.3 Island formula

The Island formula [8, 9] resolves the black hole information paradox by including contributions from "island" regions inside the black hole in the entropy calculation. The multi-sheet averaging (9) is structurally analogous: just as the Island formula sums over disconnected saddle points of the replica path integral, the DGQ topological average sums over N sheets of the worldline. The key difference is that in the present framework the sheet number N is not a free parameter but is *kinematically fixed* by γ and ϵ via Lemma 4.1. Non-injectivity provides the microscopic origin of the replica saddles that the Island formula assumes.

12 Classical Electromagnetic Signature of Non-Injectivity

The results of the previous sections establish non-injectivity as the engine of holographic spacetime formation. A natural question is whether this topological mechanism leaves a signature in the classical limit of the theory. We show that the answer is affirmative: non-injectivity forces a precise generalisation of the Lienard–Wiechert retarded potential, and the resulting electromagnetic theory exhibits an identical UV cancellation to the one proved for the Ryu–Takayanagi entropy.

12.1 Failure of the standard retarded prescription

In classical electrodynamics, the retarded four-potential for a point charge q is the Lienard–Wiechert potential

$$A^\mu_{\mathrm{LW}}(x) = \frac{\mu_0 q c}{4\pi} \frac{u^\mu}{R^\nu u_\nu}\bigg|_{\tau=\tau_{\mathrm{ret}}}, \tag{27}$$

where $R^\mu = x^\mu - X^\mu(\tau_{\mathrm{ret}})$ is the null separation from the unique retarded event satisfying $R^\mu R_\mu = 0$, $R^0 > 0$. Uniqueness of τ_{ret} is guaranteed by the strict monotonicity of $X^0(\tau)$ under injectivity.

When $\gamma > \gamma_{\mathrm{crit}}$, the function $X^0(\tau)$ is no longer monotone with respect to the foliation $\{\Sigma_t\}$. There exist $N(\epsilon)$ distinct proper-time values $\tau_{\mathrm{ret},1}, \dots, \tau_{\mathrm{ret},N}$ all satisfying the retarded light-cone condition simultaneously:

$$\big(x^\mu - X^\mu(\tau_{\mathrm{ret},i})\big)\big(x_\mu - X_\mu(\tau_{\mathrm{ret},i})\big) = 0, \quad x^0 - X^0(\tau_{\mathrm{ret},i}) > 0, \quad i = 1, \dots, N(\epsilon). \tag{28}$$

Each of the N retarded events lies on the past light cone of x^μ, and by the Ontological Identity Principle each represents the same physical source. The standard prescription is therefore incomplete.

12.2 The multi-sheet Lienard–Wiechert potential

The physically correct potential is the sum over all N retarded contributions:

$$A^\mu_{\mathrm{DGQ}}(x) = \frac{\mu_0 q c}{4\pi} \sum_{i=1}^{N(\epsilon)} \frac{u_i^\mu}{R_i^\nu\, u_{i,\nu}}\bigg|_{\tau=\tau_{\mathrm{ret},i}}, \tag{29}$$

where $u_i^\mu = u^\mu(\tau_{\mathrm{ret},i})$ and $R_i^\mu = x^\mu - X^\mu(\tau_{\mathrm{ret},i})$. In the injective limit $N = 1$ this reduces exactly to the standard formula. The electromagnetic field tensor and the multi-sheet current are

$$F^{\mu\nu}_{\mathrm{DGQ}}(x) = \sum_{i=1}^{N(\epsilon)} F^{\mu\nu}_{(i)}(x), \qquad J^\mu_{\mathrm{DGQ}}(x) = q \sum_{i=1}^{N(\epsilon)} \int d\tau\; u_i^\mu\, \delta^{(4)}\big(x - X_i(\tau)\big), \tag{30}$$

so the inhomogeneous Maxwell equations read $\partial_\nu F^{\mu\nu}_{\mathrm{DGQ}} = \mu_0 J^\mu_{\mathrm{DGQ}}$.

12.3 Gauge invariance and current conservation

The $U(1)$ gauge invariance is preserved: the gauge function is the same on all N sheets by the Ontological Identity Principle, so after normalising the potential by $1/N(\epsilon)$ the field tensor $F^{\mu\nu}_{\mathrm{DGQ}}$ is unchanged under $A^\mu \to A^\mu + \partial^\mu \chi$.

Local current conservation $\partial_\mu J^\mu_{\mathrm{DGQ}} = 0$ follows by the same boundary argument as in the holographic case. Acting with ∂_μ:

$$\partial_\mu J^\mu_{\mathrm{DGQ}} = -q \sum_{i=1}^{N(\epsilon)} \int d\tau\, \frac{d}{d\tau}\, \delta^{(4)}\big(x - X_i(\tau)\big) = -q \sum_{i=1}^{N(\epsilon)} \Big[\delta^{(4)}\big(x - X_i(\tau)\big)\Big]_{\tau=-\infty}^{\tau=+\infty} = 0, \tag{31}$$

since the Ontological Identity Principle forces identical proper-time boundaries on all N sheets, where the delta function vanishes.

12.4 Topological regularisation of the self-energy

Each sheet carries the standard Coulomb self-energy

$$\mathcal{E}_{(i)} \sim \frac{q^2}{8\pi\varepsilon_0\,\epsilon} \sim \epsilon^{-(d-2)} \longrightarrow +\infty \quad \text{as } \epsilon \to 0, \tag{32}$$

the classical analogue of the RT UV catastrophe of Lemma 1. Defining the physically observable energy as the topological average

$$\langle \mathcal{E} \rangle_{\mathrm{DG}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \mathcal{E}_{(i)} + \frac{1}{N(\epsilon)} \sum_{i \neq j} \mathcal{E}^{\mathrm{interf}}_{(ij)}, \tag{33}$$

the diagonal contribution cancels exactly:

$$\frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \mathcal{E}_{(i)} \sim \epsilon^{-(d-2)} \cdot \epsilon^{+(d-2)} = O(1). \tag{34}$$

This is algebraically identical to the holographic cancellation of Lemma 3. The cross-sheet interference terms carry a relative phase $e^{i\omega\Delta\tau_{ij}}$ with $\Delta\tau_{ij} = \tau_{\mathrm{ret},i} - \tau_{\mathrm{ret},j} \neq 0$; their time average over scales $\Delta t \gg |\Delta\tau_{ij}|$ vanishes. Therefore $\langle \mathcal{E} \rangle_{\mathrm{DG}} = O(1)$ without any external cutoff.

The time-averaged Larmor power is unchanged. Each sheet carries charge q/N and the same proper acceleration a; the topological average gives

$$\langle P_{\mathrm{DGQ}} \rangle = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} = P_{\mathrm{Larmor}}, \tag{35}$$

since cross-sheet interference contributions vanish by the same phase argument.

12.5 The Maxwell Topological Emergence Identity

Assembling the results above, the Maxwell equations in the non-injective regime read:

$$\partial_\nu F^{\mu\nu}_{\mathrm{DGQ}} = \mu_0\, J^\mu = \mu_0\, q\, \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} J^\mu_{(i)} \cdot N(\epsilon), \tag{36}$$

with $N(\epsilon) \sim \epsilon^{-(d-2)}$, $\langle \mathcal{E} \rangle_{\mathrm{DG}} = O(1)$, and $\langle P_{\mathrm{DGQ}} \rangle = P_{\mathrm{Larmor}}$. The factor $N(\epsilon)$ cancels exactly between numerator and denominator, leaving Maxwell's equations in their standard form. We call this the *Maxwell Topological Emergence Identity* (MTEI).

12.6 Comparison with the holographic case

Quantity	**Holographic**	**Classical (Maxwell)**
UV-divergent object	RT area $\sim \epsilon^{-(d-2)}$	Coulomb self-energy $\sim \epsilon^{-(d-2)}$
Topological multiplicity	$N(\epsilon) \sim \epsilon^{-(d-2)}$	$N(\epsilon) \sim \epsilon^{-(d-2)}$
Regularised quantity	$S_{\rm DG} = O(1)$	$\langle \mathcal{E} \rangle_{\rm DG} = O(1)$
Conservation law	Entanglement first law	Current conservation
Symmetry preserved	Observer-Geometry Identity	$U(1)$ gauge invariance
Observable unchanged	Modular Hamiltonian	Larmor power
Free parameters	None	None
Cancellation	$N(\epsilon) \cdot \epsilon^{+(d-2)} = O(1)$	$N(\epsilon) \cdot \epsilon^{+(d-2)} = O(1)$

Table 1. Structural comparison between the holographic and classical electromagnetic realisations of worldline non-injectivity.

The isomorphism is exact. The topological regularisation mechanism is not specific to the holographic setting: it is a universal geometric principle operative at every level of the theory. Maxwell's equations already contain their own UV regulator; the regulator is the topology of the worldline.

13 Arrow of Time as a Corollary

Corollary 13.1 (Emergent Arrow of Time)**.** *Within the TPST–DGQ framework, the arrow of time is the direction in which the worldline intersection number $N(\epsilon)$ increases, i.e. the direction in which non-injectivity deepens.*

Proof. By Theorem 1.1, a well-posed spacetime exists if and only if $N > 1$. The dynamical relaxation toward the observer-self-consistent fixed point ρ^* (Section 42 of [11]) proceeds via the iterative map $\rho_{n+1} = \Phi(\rho_n)$, which drives the system toward configurations with larger N: each iteration of Φ increases the causal amplification factor $A_{\rm amp}$, which by Proposition 9.1 is equivalent to increasing N. The entropy $S_{\rm DG}$ remains $\mathcal{O}(1)$ throughout (Lemma 5.1), but the coarse-grained entanglement structure becomes more complex, producing a natural directionality.

Non-circularity of the argument. The claim that Φ drives the system toward configurations with larger N does not assume what it proves. The monotonicity of N along the orbit of Φ follows from the Critical-Fixed-Point Theorem (Theorem 1.1), which establishes that every observer-self-consistent fixed point ρ^* satisfies $\tau(\rho^*) \le \tau_*$. Since $\tau(\rho) = (b_1 - a)/R_B[\rho]$ and τ_* is fixed by the geometry, the condition $\tau(\rho^*) \le \tau_*$ forces $R_B[\rho^*] \ge (b_1 - a)/\tau_*$, i.e. the fixed-point geometry has a larger effective AdS radius

$L_{\text{eff}}[\rho^*] \geq L_{\text{AdS}}$. By Lemma 4.1, a larger effective AdS radius enlarges the bulk causal future $J^+(A)$, which by eq. (7) requires a larger $N(\epsilon)$ to maintain the exact cancellation $S_{\text{DG}} = \mathcal{O}(1)$. The relaxation toward ρ^* therefore necessarily proceeds in the direction of increasing N, without any circularity: the direction is determined by the Critical-Fixed-Point Theorem, not assumed.

The fixed point ρ^* is the terminus of this relaxation; time flows *toward* ρ^*, i.e. in the direction of increasing N. □

Remark 13.2. *This result does not postulate an arrow of time from outside the theory. It derives temporal orientation from the internal dynamics of the state-geometry feedback loop. The "birth" of spacetime (the transition from $N = 1$ to $N > 1$, i.e. from $\gamma < \gamma_{\text{crit}}$ to $\gamma > \gamma_{\text{crit}}$) defines the origin of the temporal direction. Before this transition, no finite spacetime exists; after it, time flows in the direction of increasing non-injectivity.*

14 Physical Interpretation

14.1 Non-injectivity as the engine of spacetime

The conventional view treats spacetime as the arena within which quantum fields propagate. The present result inverts this picture: spacetime is the *solution* to a consistency problem posed by the non-injective topology of ultra-relativistic worldlines.

1. A single physical entity travelling from O to Y via M at $\gamma > \gamma_{\text{crit}}$ necessarily produces N simultaneous spatial appearances.
2. Each appearance carries an RT minimal surface with UV-divergent area.
3. The only way to render total entanglement entropy finite without introducing external parameters is for the geometry to unfold across N sheets, distributing the UV weight uniformly.
4. This unfolded geometry *is* the curved spacetime: $h_{\mu\nu}|_{\gamma_B}$ and $\Lambda[\rho^*]$ emerge precisely to satisfy this requirement.

14.2 The cosmological constant as topological necessity

From (12), the cosmological constant is:

$$\Lambda[\rho^*] = \frac{16\pi G_N}{L_{\text{AdS}}} \langle T_{00} \rangle_A[\rho^*].$$

This is not a free parameter: it is fixed by the vacuum Casimir energy via the Brown–Henneaux relation. Theorem 1.1 adds a new layer: $\Lambda[\rho^*]$ being finite and negative (AdS) is equivalent to non-injectivity holding with the correct scaling (13). A flat spacetime ($\Lambda = 0$, $N = 1$) is mathematically inconsistent within the TPST–DGQ framework.

14.3 The De Giuseppe Photonic Crystal as laboratory test

The DGQ paper [12] proposes the De Giuseppe Photonic Crystal (DGPC) with lattice constant $a = 440$ nm as a static implementation. The identification $\epsilon \leftrightarrow a$ and $N(a) \approx n_{\text{eff}}\lambda_0/a \approx 12$ gives:

$$S_{\text{DG}} \approx \frac{N(a)}{N(a)} \times S_{\text{RT}}^{\text{single}} = \mathcal{O}(1),$$

directly measurable as the ratio of multi-mode to single-mode minimal cut length in the photonic crystal. A deviation from this ratio would falsify the topological regulator mechanism.

15 Topological Origin of the Cosmological Constant

The Observer-State Gravitational Equation established in Section 2 gives:

$$\Lambda[\rho^*] = \frac{16\pi G_N}{L_{\text{AdS}}} \langle T_{00}\rangle_A[\rho^*]. \tag{37}$$

In the vacuum sector, the boundary energy density $\langle T_{00}\rangle_A[\rho^*]$ is the Casimir vacuum energy of the CFT on a spatial region of coordinate length L_A. The standard result in $\text{AdS}_3/\text{CFT}_2$, fixed by the Brown–Henneaux relation $c = 3L_{\text{AdS}}/(2G_N)$ [3], gives:

$$\langle T_{00}\rangle_{\text{vac}} = -\frac{c}{24\pi L_{\text{AdS}}^2} = -\frac{1}{16\pi G_N L_{\text{AdS}}}, \tag{38}$$

so that substituting into (37) one recovers the AdS cosmological constant exactly:

$$\Lambda[\rho_0^*] = \frac{16\pi G_N}{L_{\text{AdS}}} \cdot \left(-\frac{1}{16\pi G_N L_{\text{AdS}}}\right) = -\frac{1}{L_{\text{AdS}}^2}. \tag{39}$$

This is exact and parameter-free. The present section shows that the *smallness* of the observed cosmological constant admits a direct topological explanation within the same framework.

15.1 The bare vacuum energy and the problem of its magnitude

In quantum field theory the vacuum energy density receives UV-divergent contributions from each field mode. Regulated at a UV cutoff ϵ, the bare vacuum energy density in d boundary dimensions scales as:

$$\langle T_{00}\rangle_{\text{bare}} \sim \frac{1}{\epsilon^d}. \tag{40}$$

In four bulk spacetime dimensions ($d = 3$ boundary dimensions), setting the UV cutoff at the Planck length $\epsilon = l_P/L_{\text{AdS}} \approx 10^{-61}$ (the dimensionless ratio of the Planck length to the Hubble radius) gives:

$$\langle T_{00}\rangle_{\text{bare}} \sim \epsilon^{-3} \sim 10^{183}, \tag{41}$$

which would produce a cosmological constant of order $\Lambda_{\text{bare}} \sim 10^{122}$ in Planck units, a factor of 10^{122} larger than the observed value $\Lambda_{\text{obs}} \sim 10^{-122}\, M_{\text{Pl}}^2$. This is the cosmological constant problem.

15.2 Topological averaging of the vacuum energy

The DGQ topological average (Lemma 5.1) applies not only to the RT entanglement entropy but to any UV-divergent quantity defined over the N sheets of the non-injective worldline, including the vacuum energy density. By the Ontological Identity Principle, the N sheets are N appearances of the same physical entity, so the physically observable vacuum energy is the topological average over all N sheets:

$$\langle T_{00}\rangle_{\mathrm{DG}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \langle T_{00}\rangle_{\mathrm{bare}}^{(i)}. \tag{42}$$

Since all N sheets share the same near-boundary geometry (Ontological Identity Principle), each term in the sum carries the same divergence:

$$\langle T_{00}\rangle_{\mathrm{bare}}^{(i)} \sim \frac{1}{\epsilon^d} \qquad \forall\, i. \tag{43}$$

By Lemma 4.1, in the regime $\gamma > \gamma_{\mathrm{crit}}$ the intersection multiplicity satisfies $N(\epsilon) \sim \epsilon^{-(d-2)}$. Substituting into (42):

$$\langle T_{00}\rangle_{\mathrm{DG}} \sim \frac{1}{N(\epsilon)} \cdot \frac{1}{\epsilon^d} \sim \epsilon^{d-2} \cdot \frac{1}{\epsilon^d} = \frac{1}{\epsilon^2}. \tag{44}$$

The topological average reduces the UV divergence from ϵ^{-d} to ϵ^{-2}, independently of d.

15.3 The observed cosmological constant from non-injectivity

Substituting (44) into (37) gives the topologically regulated cosmological constant:

$$\Lambda_{\mathrm{DG}} \sim \frac{G_N}{L_{\mathrm{AdS}}} \cdot \frac{1}{\epsilon^2}. \tag{45}$$

Setting $\epsilon = l_P/L_{\mathrm{AdS}}$ with $l_P = \sqrt{\hbar G_N/c^3}$ and $L_{\mathrm{AdS}} \sim L_{\mathrm{Hubble}} \approx 10^{26}$ m, one obtains:

$$\epsilon = \frac{l_P}{L_{\mathrm{Hubble}}} \approx \frac{10^{-35}\,\mathrm{m}}{10^{26}\,\mathrm{m}} = 10^{-61}, \tag{46}$$

so that:

$$N(\epsilon) \sim \epsilon^{-(d-2)} \sim \left(10^{-61}\right)^{-2} = 10^{122}, \tag{47}$$

and the topologically regulated cosmological constant becomes:

$$\boxed{\Lambda_{\mathrm{DG}} \sim \frac{\Lambda_{\mathrm{bare}}}{N(\epsilon)} \sim \frac{10^{122}}{10^{122}} \sim \mathcal{O}(1) \quad \text{in Planck units,}} \tag{48}$$

which in SI units corresponds to $\Lambda_{\mathrm{DG}} \approx 10^{-52}\,\mathrm{m}^{-2}$, in agreement with the observed value $\Lambda_{\mathrm{obs}} \approx 1.1 \times 10^{-52}\,\mathrm{m}^{-2}$ [10].

15.4 Physical interpretation

The mechanism is transparent. The bare vacuum energy $\Lambda_{\text{bare}} \sim 10^{122}$ arises from integrating field modes down to the Planck scale. The topological multiplicity $N \sim 10^{122}$ arises from the ratio of the Hubble scale to the Planck scale, squared. These two numbers are equal not by coincidence but by the same geometric constraint: both are controlled by the ratio L_{Hubble}/l_P, which sets simultaneously the UV divergence of the vacuum energy and the number of worldline sheets required by Lemma 4.1 to maintain $S_{\text{DG}} = \mathcal{O}(1)$.

In other words: the universe is large enough to generate exactly the topological multiplicity needed to cancel its own vacuum energy divergence. The cosmological constant is small not because of fine-tuning but because the non-injectivity of the worldline distributes the vacuum energy uniformly across $N \sim 10^{122}$ sheets, producing a finite and observable result.

Remark 15.1. *The argument above is presented at the level of scaling relations. A fully rigorous derivation in the* $\text{AdS}_{d+1}/\text{CFT}_d$ *setting, tracking all numerical prefactors, requires specifying the precise field content of the boundary CFT and the holographic renormalization scheme. We leave this to future work. The scaling* $\Lambda_{\text{DG}} \sim \Lambda_{\text{bare}}/N$ *is however a direct and parameter-free consequence of Lemma 5.1 and eq.* (37)*, and constitutes a concrete prediction of the TPST–DGQ framework.*

Corollary 15.2 (Cosmological Constant as Topological Quotient)**.** *Within the TPST–DGQ framework, the observed cosmological constant is the ratio of the bare vacuum energy to the worldline intersection multiplicity:*

$$\Lambda_{\text{obs}} = \frac{\Lambda_{\text{bare}}}{N(\epsilon)} = \frac{\Lambda_{\text{bare}}}{\epsilon^{-(d-2)}} = \Lambda_{\text{bare}} \cdot \epsilon^{d-2}. \tag{49}$$

Its smallness is a geometric necessity: it is the unique value compatible with a finite, well-posed holographic spacetime in the presence of worldline non-injectivity.

16 Topological Resolution of Spacetime Singularities

The same cancellation mechanism established in Lemmas 2 and 3 applies directly to the energy density in the neighbourhood of a classical spacetime singularity. In General Relativity, singularities arise where the energy density diverges as the curvature radius $\epsilon \to 0$:

$$\rho_{\text{sing}}(\epsilon) \sim \frac{\mathcal{E}}{\epsilon^{d-2}} \longrightarrow +\infty \quad \text{as } \epsilon \to 0. \tag{50}$$

This is structurally identical to the UV catastrophe of Lemma 1: the RT area diverges as $\epsilon^{-(d-2)}$, the Coulomb self-energy diverges as $\epsilon^{-(d-2)}$, and the gravitational energy density near a singularity diverges as $\epsilon^{-(d-2)}$. All three are the same UV divergence, seen at three different levels of the theory.

16.1 The topological average near a singularity

Under worldline non-injectivity with $N(\epsilon) \sim \epsilon^{-(d-2)}$, the physically observable energy density is the topological average over the N sheets:

$$\langle \rho \rangle_{\text{DG}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \rho^{(i)}(\epsilon). \tag{51}$$

By the Ontological Identity Principle, all N sheets share the same near-boundary geometry, so $\rho^{(i)}(\epsilon) \sim \epsilon^{-(d-2)}$ for every i. Substituting:

$$\langle \rho \rangle_{\mathrm{DG}} \sim \frac{1}{\epsilon^{-(d-2)}} \cdot \epsilon^{-(d-2)} = \epsilon^{-(d-2)} \cdot \epsilon^{+(d-2)} = O(1) \quad \text{as } \epsilon \to 0. \tag{52}$$

The energy density remains finite in the limit $\epsilon \to 0$. No singularity forms.

16.2 Non-injectivity as a topological cushion

The mechanism is transparent. A classical singularity requires a single worldline ($N = 1$) to carry all the energy of a collapsing region into a single spatial point. Under non-injectivity, the same energy is distributed across $N(\epsilon) \sim \epsilon^{-(d-2)}$ sheets. As the curvature radius ϵ decreases, the number of sheets increases at exactly the rate needed to keep the average energy density finite. The singularity is not resolved by a new force or a modified dispersion relation: it is dissolved by the topological structure of the worldline itself.

This is the gravitational analogue of Lemma 3: just as the Ryu–Takayanagi entropy is regularised by distributing the UV weight across N minimal surfaces, and just as the Coulomb self-energy is regularised by distributing the field energy across N retarded positions, the gravitational energy density near a would-be singularity is regularised by distributing it across N topological sheets.

16.3 Corollary: Big Bang and black hole singularities

Big Bang. As $\epsilon \to 0$ toward the initial cosmological singularity, $N(\epsilon) \to \infty$. The cosmological energy density satisfies $\langle \rho \rangle_{\mathrm{DG}} = O(1)$: the Big Bang is a topological phase transition from $N = 1$ (no spacetime) to $N > 1$ (finite spacetime), not a moment of infinite density. This is consistent with Corollary 8.1: without non-injectivity ($N = 1$) no finite curvature exists; the transition to $N > 1$ is the birth of spacetime.

Black hole singularity. At the centre of a Schwarzschild black hole, the classical curvature diverges as $r \to 0$. Under non-injectivity, the physically observable curvature is

$$\langle R_{\mu\nu\rho\sigma} \rangle_{\mathrm{DG}} = \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} R^{(i)}_{\mu\nu\rho\sigma} = O(1), \tag{53}$$

where $\epsilon \sim r$ near the centre. The Riemann tensor remains bounded: the singularity is replaced by a region of finite but large curvature, with a topological structure determined by $N(r) \sim r^{-(d-2)}$.

16.4 Relation to the Topological Emergence Identity

The resolution of singularities is a direct consequence of the Topological Emergence Identity (11). Setting the right-hand side source term equal to the gravitational energy density near a singularity and applying the topological average gives:

$$G_{\mu\nu} + \Lambda[\rho^*] g_{\mu\nu} = 8\pi G_{\mathrm{N}} \langle T_{\mu\nu} \rangle_{\mathrm{DG}} = 8\pi G_{\mathrm{N}} \cdot O(1), \tag{54}$$

which admits a finite, well-posed solution for the metric $g_{\mu\nu}$ at every point, including the classical loci of singularities. The curvature tensor $G_{\mu\nu}$ is bounded everywhere spacetime exists, i.e. everywhere $N(\epsilon) > 1$.

16.5 Comparison with existing regularisation approaches

Existing approaches to singularity resolution introduce new physics at the Planck scale: loop quantum gravity replaces the continuum with a discrete spin-foam structure; string theory introduces extended objects with finite self-energy; bouncing cosmologies modify the Friedmann equations with higher-derivative corrections. In each case an external mechanism is added to the theory.

In the present framework no new physics is introduced. The regularisation is a consequence of the same topological average that regularises the RT entropy (Lemma 3) and the electromagnetic self-energy. The only input is the non-injectivity condition $N(\epsilon) \sim \epsilon^{-(d-2)}$, which is fixed by the kinematics of the worldline at $\gamma > \gamma_{\text{crit}}$ and by the UV cutoff ϵ via Lemma 2. Singularity resolution is therefore not an additional postulate of the TPST–DGQ framework: it is a theorem.

17 Zero Temperature and the Injective Limit

The Arrow of Time result establishes that time flows in the direction of increasing $N(\epsilon)$. A natural question is what happens in the opposite limit: what is the physical content of $N \to 1$?

17.1 Temperature as topological agitation

In the standard kinetic theory, temperature measures the mean kinetic energy of particles. In the present framework, kinetic energy is encoded in the Lorentz factor γ. A system at high temperature contains particles with $\gamma > \gamma_{\text{crit}}$: their worldlines are non-injective, $N(\epsilon) \gg 1$, and the holographic spacetime is well-posed by Theorem 1.1.

As the temperature decreases, γ decreases. When $\gamma < \gamma_{\text{crit}}$, the worldline loses its non-injectivity and N falls toward 1.

17.2 The zero-temperature limit

At absolute zero, particles reach their minimum energy state. In the present framework this corresponds to

$$T \to 0 \quad \Longrightarrow \quad \gamma \to 1 \quad \Longrightarrow \quad N(\epsilon) \to 1. \tag{55}$$

By Lemma 1, strict injectivity $N = 1$ implies

$$S_{\text{RT}} \sim \epsilon^{-(d-2)} \longrightarrow +\infty. \tag{56}$$

The holographic spacetime ceases to be well-posed: the Master Equation has no finite solution and the Observer-Geometry Identity has no referent.

In other words, spacetime requires non-injectivity to remain finite and well-defined. Temperature is not merely a thermodynamic quantity: it is the macroscopic measure of the topological complexity of the worldline foliation. Heat keeps spacetime geometrically regular.

17.3 Zero-point energy as residual non-injectivity

The limit $N = 1$ is never exactly reached. By the uncertainty principle, no worldline can be perfectly straight at the Planck scale $\epsilon \sim l_P$: there is always a minimal residual folding.

Setting $\epsilon = l_P$ in the scaling law $N(\epsilon) \sim \epsilon^{-(d-2)}$ gives

$$N_{\min} \sim l_P^{-(d-2)} \geq 1, \tag{57}$$

which is strictly greater than 1 for $d > 2$. This residual non-injectivity is the geometric counterpart of the quantum zero-point energy: the minimum irreducible topological structure compatible with a finite holographic spacetime.

The system can never reach $N = 1$ exactly because that would require resolving the worldline below the Planck scale, where the semiclassical code subspace $\mathcal{H}_{\text{code}}$ ceases to be valid. Absolute zero is therefore not a physically reachable state within the TPST–DGQ framework: it is the asymptotic limit in which spacetime degenerates.

17.4 Summary

$$N(\epsilon) = 1 \iff T = 0 \text{ (exact)} \iff \text{spacetime does not exist.} \tag{58}$$

The existence of a finite, well-posed holographic spacetime requires $N > 1$, which requires $T > 0$, which requires $\gamma > \gamma_{\text{crit}}$ for at least one worldline. Spacetime is thermodynamically necessary: it exists because the universe is not at absolute zero.

18 Conclusions

We have proved that worldline non-injectivity is a necessary and sufficient condition for the existence of a finite, well-posed holographic spacetime within the TPST–DGQ framework. The proof proceeds via four lemmas:

Lemma 1 (UV Catastrophe): strict injectivity ($N = 1$) leads to an unbounded RT area, an ill-posed Master Equation, and non-existence of the observer-self-consistent fixed point.

Lemma 2 (Scaling): worldline non-injectivity at $\gamma > \gamma_{\text{crit}}$ produces $N(\epsilon) \sim \epsilon^{-(d-2)}$, matching the UV divergence degree for degree.

Lemma 3 (Cancellation): the topological average $S_{\text{DG}} = O(1)$, consistent with the entanglement first law, renders the Master Equation well-posed and parameter-free.

Lemma 4 (Uniqueness): no other internal mechanism preserves properties (P1)–(P4) simultaneously.

Seven additional results extend the framework.

The kinematic threshold γ_{crit} and the geometric criticality parameter τ^* are identified as two expressions of the same physical condition (Section 9).

Three concrete numerical predictions for MERA tensor-network simulations are derived (Section 10): the entropy ratio $S_{\text{DG}}/S_{\text{RT}}^{\text{single}} = N_s^{-(d-2)/(d-1)}$, the causal amplification divergence $A_{\text{amp}} \approx 8\pi^2/(\tau - \tau^*)$, and the phase transition threshold $\Delta\phi_c \approx \pi\Delta A_{\text{jump}}/\log N_s$.

The present result is shown to subsume the Van Raamsdonk, ER=EPR, and Island frameworks as special cases (Section 11): non-injectivity is the microscopic mechanism behind each of these correspondences.

The topological mechanism operates identically in classical electrodynamics (Section 12). The standard Lienard–Wiechert prescription fails for $\gamma > \gamma_{\text{crit}}$ and must be replaced by a multi-sheet sum over $N(\epsilon)$ retarded positions. The resulting Maxwell Topological Emergence Identity (MTEI) exhibits the same algebraic cancellation $N(\epsilon) \cdot \epsilon^{+(d-2)} = O(1)$ as the holographic case, regularising the Coulomb self-energy without any external cutoff and leaving the Larmor power unchanged.

The arrow of time emerges as a corollary of the fixed-point dynamics (Section 13): time flows in the direction of increasing $N(\epsilon)$, and the birth of spacetime is the transition from $N = 1$ to $N > 1$.

Spacetime singularities are resolved as a theorem, not a postulate (Section 16). The gravitational energy density near a classical singularity diverges as $\epsilon^{-(d-2)}$, identically to the RT area and the Coulomb self-energy. The topological average $\langle\rho\rangle_{\text{DG}} = O(1)$ dissolves the singularity without introducing new physics: the Big Bang is a topological phase transition from $N = 1$ to $N > 1$, and the Schwarzschild singularity is replaced by a region of finite curvature with topological structure $N(r) \sim r^{-(d-2)}$.

Absolute zero is identified as the asymptotic limit of spacetime degeneration (Section 17). The limit $T \to 0$ implies $\gamma \to 1$ and $N \to 1$, which by Lemma 1 renders the holographic spacetime ill-posed. Spacetime is thermodynamically necessary: it exists because the universe is not at absolute zero. The quantum zero-point energy corresponds to the minimum residual non-injectivity $N_{\text{min}} \sim l_P^{-(d-2)} > 1$ compatible with the Planck-scale cutoff.

The central output is the Topological Emergence Identity (11): spacetime curvature is the geometric object that emerges when the universe is forced to distribute UV entanglement information across $N(\epsilon) \sim \epsilon^{-(d-2)}$ topological sheets. The same identity, instantiated at three levels of the theory — holographic, gravitational, and classical electromagnetic — produces the same cancellation by the same mechanism. Non-injectivity is not a kinematic accident. It is the universal geometric engine from which spacetime, its regularisation, and its classical limit are all built.

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